Quantum Mechanics Technology – Angular momentum and perturbation theory

PHYS598MAP

IANAT

Note Title

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Angular momentum

Angular momentum operators are defined in quantum mechanics by their group properties (which includes how they transform under rotations)

$$[J^2, J_k] = 0$$
$$[J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k$$

really defines angular momentum operators – any operator that satisfies these commutation operators will transform a certain way under rotations

Eigenstates:

$|jm\rangle$ with $m = -j \dots j$ are eigenstates

$$J^2 |jm\rangle = \hbar^2 j(j+1)\, |jm\rangle$$
$$J_z |jm\rangle = \hbar m\, |jm\rangle$$

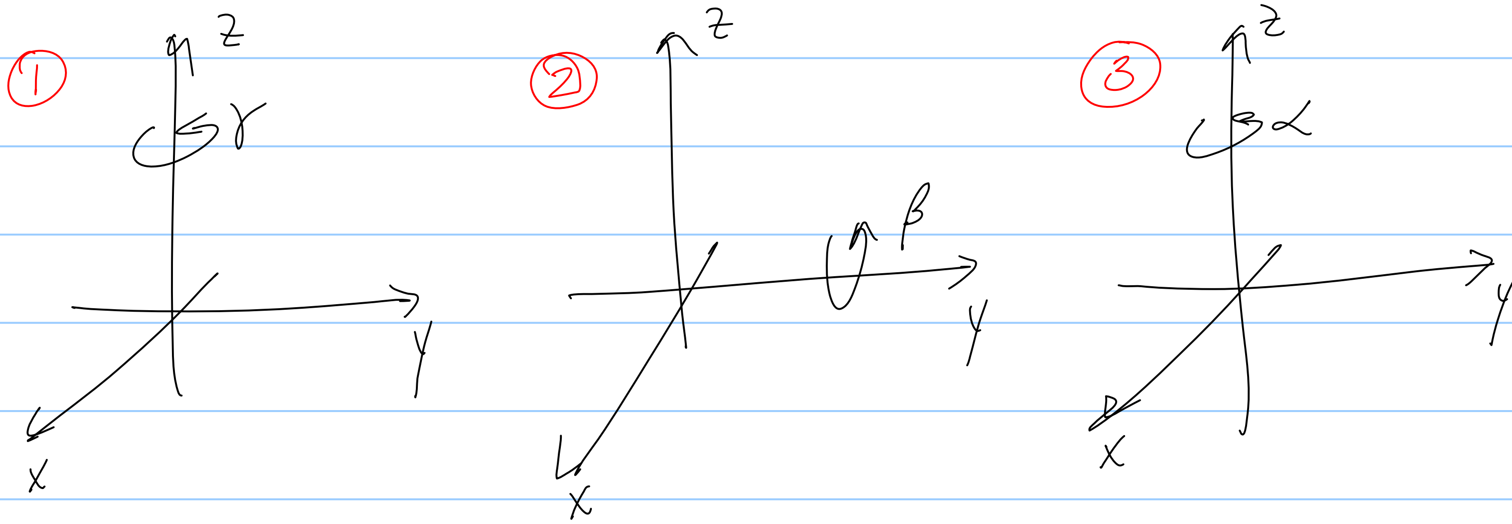
raising and lowering operators: $J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}\;|j, m+1\rangle$

$$J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\;|j, m-1\rangle$$

also: $J_x = \frac{1}{2}(J_+ + J_-)$ $\quad J_y = -\frac{i}{2}(J_+ - J_-)$

A little bit about rotations:

Use Euler angles: any rotation can be written as three rotations in succession



$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\,\mathcal{D}_y(\beta)\,\mathcal{D}_z(\gamma)$$ rotation operator !

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle jm' | e^{-i\frac{\vec{J}\cdot\hat{n}\phi}{\hbar}} | jm\rangle$$ matrix elements

note that $[J^2, \mathcal{D}] = 0$, so rotations cannot change j — rotations just mix up the m's, or:

$$\mathcal{D}(\alpha,\beta,\gamma)\,|j,m\rangle = \sum |jm'\rangle\, \mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma)$$

to get these matrix elements:

$$\mathcal{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle jm' | e^{-i\frac{J_z\alpha}{\hbar}} e^{-i\frac{J_y\beta}{\hbar}} e^{-i\frac{J_z\gamma}{\hbar}} | jm\rangle =$$

$$e^{-i(m'\alpha + m\gamma)} \underbrace{\langle jm' | e^{-iJ_y\beta/\hbar} | jm\rangle}_{\text{not trivial!}}$$

<u>define:</u> $d^{(j)}_{m'm}(\beta) = \langle jm' | e^{-iJ_y\beta/\hbar} | jm\rangle$

there are formulas and charts for these things

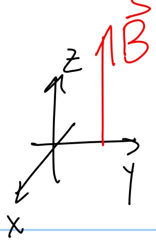
Any vector operator in quantum mechanics must transform like

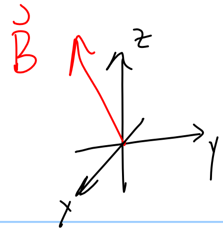
$$\vec{V} \rightarrow \mathcal{D}^{\dagger}(\alpha,\beta,\gamma)\,\vec{V}\,\mathcal{D}(\alpha,\beta,\gamma)$$

which means that $\boxed{[V_i, J_j] = i\varepsilon_{ijk}\hbar V_k}$

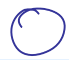
defines vector operators

Is this useful? Yes! Think ~~about~~ a Stern-Gerlach-like experiment:





sudden rotation of magnetic field



atom in state $|jm\rangle$

what's the probability distribution among m-states?

$$\psi \rightarrow \sum_{m'} \mathcal{D}^{(j)}_{mm'}(\alpha,\beta,\gamma)\,|jm'\rangle$$

$$|\psi|^2 \rightarrow \sum_{m'} \underbrace{\left|d^{(j)}_{mm'}(\beta)\right|^2}$$

probability to find atom in $|jm'\rangle$ — note that these states still refer to projection along $\hat{z}$

Angular momentum in real space

$$\vec{L} = \vec{r} \times \vec{p}$$

in spherical coordinates:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)\right]$$

$L_z = -i\hbar\frac{\partial}{\partial\varphi}$ $L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$L_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$

turns out that $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$, so we call this angular momentum

$\langle\hat{n}|lm\rangle = Y_{lm}(\theta,\varphi)$ spherical harmonics are eigenkets

orthonormal – $\int d\Omega\, Y_{lm}^*(\Omega)\, Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$

parity – $P|lm\rangle = (-1)^l|lm\rangle$ (where P reflects through the origin: $\theta\to\pi-\theta$, $\varphi\to\pi+\varphi$)

can look them up in a table, or Mathematica knows them (use SphericalHarmonicY[l, m, θ, φ])

note that $Y_{lm}(\Omega) \longrightarrow \sum_{m'} Y_{lm'}(\Omega)\, \mathcal{D}^{(l)}_{mm'}(\alpha,\beta,\gamma)$ under rotations

$$Y_\ell^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

No m-dependence for probabilities

Table 2.1 The first few spherical harmonics $Y_{lm}(\theta, \phi)$.

l	m	Spherical harmonic $Y_{lm}(\theta, \phi)$
0	0	$Y_{0,0} = \frac{1}{(4\pi)^{1/2}}$
1	0	$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$
	±1	$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{\pm i\phi}$
2	0	$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$
	±1	$Y_{2,\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \boxed{\sin\theta\cos\theta}\, e^{\pm i\phi}$
	±2	$Y_{2,\pm2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{\pm 2i\phi}$
3	0	$Y_{3,0} = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$
	±1	$Y_{3,\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1)\, e^{\pm i\phi}$
	±2	$Y_{3,\pm2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta\, e^{\pm 2i\phi}$
	±3	$Y_{3,\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta\, e^{\pm 3i\phi}$

θ

ϕ

they all look this way — separable in θ and ϕ

Spin-1/2 system

A very important example, because

$\boxed{\text{all 2-level quantum systems are the same}}$

denote eigenkets as $|+\rangle$ and $|-\rangle$, or $|\uparrow\rangle$ and $|\downarrow\rangle$, or $|0\rangle$ and $|1\rangle$

state vector: $\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$

Components of $\vec{S}$:

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -| \right)$$

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +| \right)$$

$$S_y = \frac{\hbar}{2}\left(-|+\rangle\langle -| + |-\rangle\langle +| \right)$$

Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>expectation values</u>: $\langle S_k \rangle = \frac{\hbar}{2} \chi^\dagger \sigma_k \chi$

<u>rotation operator</u>: $e^{-i\vec{\sigma}\cdot\hat{n}\,\varphi/2}$ $\qquad (\vec{\sigma}\cdot\hat{n})^n = 1$ n even
$\qquad\qquad = \vec{\sigma}\cdot\hat{n}$ n odd

$$= \begin{pmatrix} \cos\varphi/2 - in_z \sin(\varphi/2) & (-in_x - n_y)\sin\varphi/2 \\ (-in_x + n_y)\sin\varphi/2 & \cos\varphi/2 + in_z \sin\varphi/2 \end{pmatrix}$$

note! $e^{-i\vec{\sigma}\cdot\hat{n}\,\varphi/2}\Big|_{\varphi=2\pi} = -1$ for any $\hat{n}$

"4π" rotation required to get back to the original state

<u>Angular momentum addition</u>

$\rightarrow$ <u>2-vectors</u>: $\vec{J} = \vec{J}_1 + \vec{J}_2$

new eigenkets $|j_1 j_2; j, m\rangle$
with $j = (j_1 + j_2) \ldots (j_1 - j_2)$ — note that
j_1 and j_2 are still good quantum numbers

to rewrite new eigenkets in terms of the old ones:

$$|j_1 j_2; j m\rangle = \sum_{\substack{m_1 \\ m_2}} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle}$$

Clebsch-Gordan coefficient

0 unless: $m = m_1 + m_2$

$|j_1 - j_2| \leq j \leq j_1 + j_2$

"triangle inequality"

Mathematica knows these:
(use ClebschGordan[{j_1, m_1}, {j_2, m_2}, {j, m}])
and there is a handy table (see handout and class notes on the website)

Sometimes you'll also see 3-j symbols:

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \underbrace{\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}}_{\text{3-j symbol}}$$

there are different conventions about this factor important example for addition: spin-$\frac{1}{2}$

2 spin-$\frac{1}{2}$ vectors $\longrightarrow$ singlet state: $\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ $s=m=0$

triplet states: $s=1$

$$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \quad |++\rangle, \quad |--\rangle$$

$m=0 \qquad m=+1 \qquad m=-1$

Addition of 3-angular momenta:

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$$

if we just want a representation where J^2 and J_z are diagonal, there are 3 ways to constructing basis states:

add two vectors to make $\vec{J}'$, then add the third to make $\vec{J}$

ex: $\vec{J}' = \vec{J}_1 + \vec{J}_2$

$\vec{J} = \vec{J}' + \vec{J}_3$

- the "right way" to do this depends on the particular problem (we'll see examples), and is usually set by relative energy scales

how do we connect different representations (the different ways to add the vectors together)?

we invent new coefficients!

<u>ex:</u> $\vec{J}' = \vec{J}_1 + \vec{J}_2$ $\qquad$ $\vec{J}'' = \vec{J}_2 + \vec{J}_3$

build eigenstates of $\vec{J}$ by adding $\vec{J}_3 + \vec{J}' = \vec{J}$ to make $|(j_1 j_2) j_3 ; j m\rangle$ or $\vec{J}'' + \vec{J}_1 = \vec{J}$ to make $|j_1 (j_2 j_3) ; j m\rangle$

so: $j' = |j_1 - j_2| \dots |j_1 + j_2|$ $\qquad$ $j'' = |j_2 - j_3| \dots |j_2 + j_3|$

$$|(j_1 j_2) j_3 ; j m\rangle = \sum_{j''} R_{j'' j'} \, |j_1 (j_2 j_3) ; j m\rangle$$

related to Racah coefficients:

$$R_{j'' j'} = \sqrt{(2j''+1)(2j'+1)} \; \underbrace{W(j_1 j_2 j_3 ; j j' j'')}_{\text{Racah coefficent}}$$

Mathematica knows 6-j symbols, which are related to Racah coefficients:

$$\underset{\text{6-j symbol}}{\longrightarrow}\begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j' & j'' \end{Bmatrix} = (-1)^{j_1+j_2+j+j_3}\, W(j_1 j_2 j j_3; j' j'')$$

use $\text{SixJSymbol}[\underbrace{\{a,b,c\}}_{\text{1st row}}, \underbrace{\{d,e,f\}}_{\text{2nd row}}]$
in Mathematica

A cooked-up example: a particle of spin 3/2, initially at rest, decays into particle a with spin 1/2 and particle b with spin 0

① What are possible values of the relative orbital angular momentum after decay? What is the parity of the possible final states?

Must have $\vec{J} = \vec{S}_a + \vec{S}_b + \vec{L}$ — conservation of angular momentum

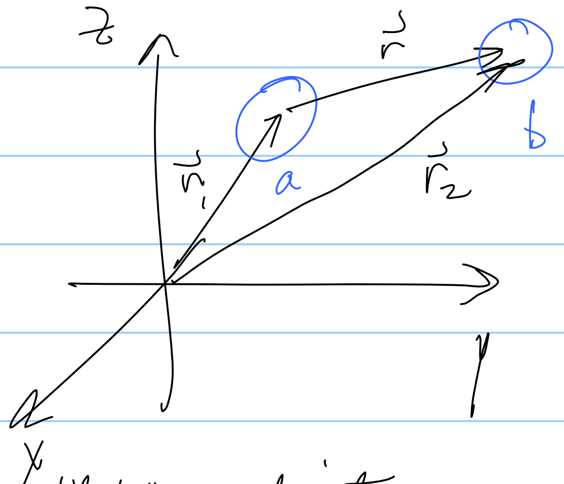
This is a boring example with three angular momenta — only one way to add them, since $\vec{S}_b = 0$

We must have $l = 1, 2$, so that we can add $l + S_a$ and get $j = 3/2$ $\left(\tfrac{1}{2} + 1 = \tfrac{3}{2},\ 2 - \tfrac{1}{2} = \tfrac{3}{2}\right)$

The parity of these states is $(-1)^l$ — there's one of even and one of odd parity

② What coordinate does the orbital angular momentum refer to? What's happening in real space?

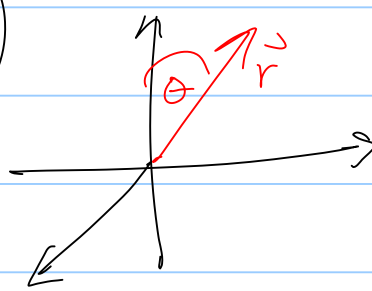
$$|\psi\rangle = |R\rangle\,|lm\rangle$$



"lab" coordinate system

$\vec{r} = \vec{r}_1 - \vec{r}_2$ relative coordinate

$Y_{lm}(\theta, \varphi)$ — angles refer to orientation of $\vec{r}$ relative to (x, y, z)



parity tells you about how the wavefunction changes under exchange

$|Y_{\ell m}(\theta,\varphi)|^2$ tells you the probability of finding either particle in direction (θ,φ) — the other particle is always perfectly correlated by $\hat{r}$

Irreducible tensors

Operators that transform under rotation like spherical harmonics, or vectors, or ...

$T_q^{(k)}$ spherical basis notation — "irreducible" means that this tensor transforms just like a spherical harmonic with $l=k$ and $m=q$ — nothing else is mixed in

must have: $\mathcal{D}(\alpha,\beta,\gamma)\, T_q^{(k)}\, \mathcal{D}^\dagger(\alpha,\beta,\gamma) = \sum_{q'=-k}^{k} \mathcal{D}_{qq'}^{(k)}\, T_{q'}^{(k)}$

$$[J_z, T_q^{(k)}] = \hbar q\, T_q^{(k)}$$
$$[J_\pm, T_q^{(k)}] = \hbar \sqrt{(k \mp q)(k \pm q + 1)}\; T_{q\pm 1}^{(k)}$$

defines irreduceable tensor

Don't think of irreducedble tensor as a matrix — then you've chosen a representation. These tensors are just defined in terms of their transformation properties.

spherical basis defined by:

$$\hat{e}_0 = \hat{z} \qquad \hat{e}_1 = -\tfrac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \qquad \hat{e}_{-1} = \tfrac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$$

so, any vector can be written as an irreduceable tensor!

$$\vec{A} = |A| \sqrt{\frac{4\pi}{3}} \sum_q Y_{1q}\, \hat{e}_q^{\,*}$$

$$\boxed{\begin{aligned} A_0 &= A_z \\ A_1 &= -\tfrac{1}{\sqrt{2}}(A_x + iA_y) \\ A_{-1} &= \tfrac{1}{\sqrt{2}}(A_x - iA_y) \end{aligned}}$$

recall:

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$
$$Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}$$

We'll run into products of 2 tensors:

$$A^{(k)} \cdot B^{(k)} = \sum A_q B_{-q} (-1)^q \quad \text{scalar product}$$

more general product is called a "contraction" : $\left(T^{(k_1)} \otimes T^{(k_2)}\right)_q^{(k)}$

$$\left(T^{(k_1)} \otimes T^{(k_2)}\right)_q^{(k)} = \sum_{\substack{q_1 \\ q_2}} \underbrace{\langle k_1 k_2 ; k q | k_1 k_2 ; q_1 q_2 \rangle}_{\text{regular rules apply!}} \; T_{q_1}^{(k)} T_{q_2}^{(k)}$$

regular rules apply! $q = q_1 + q_2$
$|k_1 - k_2| \le k \le k_1 + k_2$

<u>example:</u> $\left(A^{(1)} \otimes B^{(1)}\right)_0^{(1)} =$

$$0 \cdot A_0 B_0 + \frac{1}{\sqrt{2}} A_1 B_{-1} - 1/\sqrt{2}\, A_{-1} B_1$$

$$= 1/\sqrt{2} \left(A_1 B_{-1} - A_{-1} B_1\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left[-(A_x + iA_y)(B_x - iB_y) + (A_x - iA_y)(B_x + iB_y) \right]$$

$$= \frac{2}{2\sqrt{2}} \left[-iA_y B_x + i A_x B_y \right]$$

$$= \frac{i}{\sqrt{2}} \left[A_x B_y - A_y B_x \right] = \frac{i}{\sqrt{2}} \left(\vec{A} \times \vec{B}\right)_z$$

$\boxed{\text{Wigner–Eckart theorem}}$

tells you how to get the matrix element of a spherical tensor

$$\langle \alpha j' m' | T^{(k)}_q | \alpha j m \rangle = \underbrace{\langle j k; j' m' | j k; m q \rangle}_{\text{this is for coupling } j \text{ and } k \text{ to make } j'} \frac{\langle \alpha j' \| T^{(k)} \| \alpha j \rangle}{\underbrace{\sqrt{2j+1}}_{\text{there are different conventions for this}}}$$

α: represents all quantum numbers besides angular momentum

$\langle \| \, \| \rangle$ – reduced matrix element – note that it's independent q, m, m'

<u>What's it good for?</u>

- finding matrix elements – get it for one set of q, m, and m', then use Clebsch–Gordan coefficients to generalize
- selection rules!

  let's say that $T^{(k)}_q$ is the interaction part of a Hamiltonian – then the Wigner–Eckart theorem tells you which states can be coupled

$k=0$ - scalar - $j=j'$, $m=m'$

$k=1$ - vector - $\Delta m = \pm 1, 0$ $\Delta j = \pm 1, 0$

for vector operators there is an additional selection rule that's worth mentioning when the angular momentum states are real space states

$$\langle l'm' | A_q | lm \rangle = |A| \sqrt{\frac{4\pi}{3}} \langle l'm' | Y_1^q | lm \rangle$$

$$= |A| \sqrt{\frac{4\pi}{3}} \int d\Omega \, Y_{l'}^{m'*} Y_1^q Y_l^m$$

$$\propto \langle 1 \, l \, ; 0 0 | 1 \, l \, ; l' 0 \rangle \underbrace{\langle 1 \, l \, ; q m | 1 \, l \, ; l' m' \rangle}_{}$$

adding Y_1 + l to make l' $\rightarrow$ $\boxed{l' = l \pm 1}$

"parity selection rule"

<u>Practical example:</u> nuclear physics – there is a very similar problem in atomic physics that we'll get to!

$^{49}_{21}Sc$ nucleus – describe as a spinless core of 20 protons and 28 neutrons, with a single valence proton orbiting

let's say total valence angular momentum $j = 3/2$
($\vec{J} = \vec{L} + \vec{S}$, w/ $s = 1/2$)

x-ray hits nucleus, and the proton absorbs it – matrix element is $\langle j' m' | \vec{r} | 3/2, m \rangle$ (an electric dipole transition – more about that later in the course!)

① What's angular momentum of the proton after absorption?

$j = 5/2, 3/2, 1/2$ (triangle inequality, w/ $T_q^{(k)} = \vec{r}$, so $k = 1$)

② This is what's neat — if $j' = 3/2$, what are the relative probabilities for different m'?

Wigner-Eckart theorem — just compare Clebsch-Gordan coefficients squared

let's say $m = 1/2$, then $m' = -1/2, 1/2, 3/2$

there are the relative probabilities

$$|\langle 1, 3/2; 3/2, 3/2 | 1, 3/2; 1, 1/2\rangle|^2 = 2/5$$

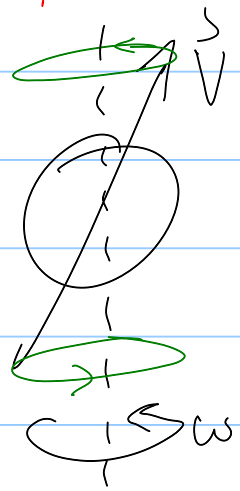
$$|\langle 1, 3/2; 3/2, 1/2 | 1, 3/2; 0, 1/2\rangle|^2 = 1/15$$

$$|\langle \underset{k}{1}, \underset{j}{3/2}; \underset{j'}{3/2}, \underset{m'}{-1/2} | \underset{k}{1}, \underset{j}{3/2}; \underset{q}{-1}, \underset{m}{1/2}\rangle|^2 = 8/15$$ ← most likely to do this, if q is not fixed

The projection theorem: can be proven using Wigner-Eckart theorem — also useful for understanding concept behind Wigner-Eckart theorem for a vector operator $\vec{V}$:

$$\underbrace{\langle \alpha' j m' | V_q | \alpha j m \rangle}_{\text{matrix element of } \vec{V}} = \frac{\langle \alpha' j m' | \vec{J} \cdot \vec{V} | \alpha j m \rangle}{\hbar^2 j(j+1)} \underbrace{\langle j m' | J_q | j m \rangle}_{\text{matrix element of } \vec{J}}$$

these are proportional!



rotating system (rotating at angular velocity ω)

→ the average value of $\vec{V}$ is along the rotation axis ($\vec{J}$)

This is very useful!

See homework problem on Zeeman effect in Hydrogen

We will run into matrix elements of tensor products:

$$X_q^{(k)} = \left(T^{(k_1)} \otimes T^{(k_2)}\right)_q^{(k)}$$

only operates in space 1

only in space 2

— like spin and angular momentum! $\vec{J} = \vec{L} + \vec{S}$

$\vec{J}_1$ $\vec{J}_2$

need 2 formulas:

regular Wigner-Eckart:

$$\langle \alpha j_1 j_2 j | X_q^{(k)} | \alpha' j_1' j_2' j' \rangle = (-1)^{j-m} \begin{pmatrix} j & k & j' \\ -m & q & m' \end{pmatrix} \langle \alpha j_1 j_2 j \| X^{(k)} \| \alpha' j_1' j_2' j' \rangle$$

OMG! 9-j symbol!

and:

$$\langle \alpha j_1 j_2 j \| X^{(k)} \| \alpha' j_1' j_2' j' \rangle = \sqrt{(2j+1)(2j'+1)(2k+1)} \begin{Bmatrix} j_1 & j_1' & k_1 \\ j_2 & j_2' & k_2 \\ j & j' & k \end{Bmatrix} \cdot$$

$$\sum_{\alpha''} \underbrace{\langle \alpha j_1 \| T^{(k)} \| \alpha'' j_1' \rangle \langle \alpha'' j_2 \| U^{(k)} \| \alpha' j_2' \rangle}_{\text{individual reduced matrix elements}}$$

Specific, important example:

scalar product

$$\langle \alpha j_1 j_2 j m | T^{(k)} \cdot U^{(k)} | \alpha' j_1' j_2' j' m' \rangle = (-1)^{j_1' + 2k + j_2 + 2j - m} \cdot$$

$$\begin{pmatrix} j & 0 & j \\ -m & 0 & m \end{pmatrix} \sqrt{2j+1}\; \delta_{jj'} \delta_{mm'} \begin{Bmatrix} j_1 & j_2 & j \\ j_2' & j_1' & k \end{Bmatrix} \cdot$$

$$\sum_{\alpha''} \langle \alpha j_1 \| T^{(k)} \| \alpha'' j_1' \rangle \langle \alpha'' j_2 \| U^{(k)} \| \alpha' j_2' \rangle$$

and $\langle \alpha j_1 j_2 j \| T^{(k)} \| \alpha' j_1' j_2' j' \rangle = \delta_{j_2 j_2'} (-1)^{j_1+j_2+j'+k} \sqrt{(2j+1)(2j'+1)} \cdot$

$$\begin{Bmatrix} j_1 & j & j_2 \\ j' & j_1' & k_1 \end{Bmatrix} \langle \alpha j_1 \| T^{(k)} \| \alpha' j_1' \rangle \quad \text{(switch } j_1 \text{ \& } j_2 \text{ for } U^{(k)}\text{)}$$

This is also a related, important trick:

$$\langle n'l's'j'm' | T_q^{(k)} | nlsjm \rangle = (-1)^{j'-m'} \langle n'l's'j' \| T^{(k)} \| nlsj \rangle \cdot \begin{pmatrix} j' & k & j \\ -m' & q & m \end{pmatrix}$$

if $[T_q^{(k)}, \vec{S}] = 0$, then:

$$\langle n'l's'j' \| T^{(k)} \| nlsj \rangle = (-1)^{l'+s'+j+k} \langle n'l' \| T^{(k)} \| nl \rangle \cdot \sqrt{(2j+1)(2j'+1)} \cdot \begin{Bmatrix} l' & j' & s' \\ j & l & k \end{Bmatrix}$$

Perturbation theory – quick reminder

Time independent –

$$H = H_0 + H'$$

eigenkets of H_0

non-degenerate: $\Delta E^{(1)} = \langle \psi^{(0)} | H' | \psi^{(0)} \rangle$

$$\Delta E^{(2)} = \sum_{k \neq n} \frac{\langle \psi_n^{(0)} | H' | \psi_k^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$$

$$|\psi_k\rangle = |\psi_k^{(0)}\rangle + \sum_{k \neq n} |\psi_k^{(0)}\rangle \frac{\langle \psi_k | H' | \psi_n \rangle}{E_n^{(0)} - E_k^{(0)}}$$

degenerate:

choose basis kets to diagonalize H'

→ eigenvalues are 1st order energy shifts

→ for higher order, don't include degenerate states in the sum time-dependent: $H = H_0 + V(t)$

Fermi's Golden rule:

density of states

$$\underbrace{w_{i\to f}}_{\text{transition rate from } |i\rangle \to |f\rangle} = \frac{2\pi}{\hbar} |V_{if}|^2 \int dE_n \, \overbrace{\rho(E_n)}^{\text{density of states}} \, \underbrace{\delta(E_n - E_i)}_{\text{energy conservation}}$$

energy conservation

transition rate from $|i\rangle \to |f\rangle$